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1995 J. Phys. A: Math. Gen. 28 6363

(http://iopscience.iop.org/0305-4470/28/22/012)

J. Phys. A: Math. Gen. 28 (1995) 6363–6371. Printed in the UK

A remark on symmetry of stochastic dynamical systems and their conserved quantities

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Received 24 April 1995

Abstract. The symmetry properties of stochastic dynamical systems described by a stochastic differential equation of Stratonovich type and related conserved quantities are discussed, *extending previous results by Misawa. New conserved quantities are given by applying* symmetry operators to known conserved quantities. Some detailed examples are presented.

Symmetries and conserved quantities have been discussed in the framework of Bismut's stochastic mechanics [1] and Nelson's stochastic mechanics, see, e.g., [2–4]. More recently Cruzeiro *et al* [5] and Nagasawa [6] have discussed stochastic variational principles and associated conserved quantities in the theory of Schrödinger processes (Euclidean quantum theory, in the sense of Zambrini, see also [7]). In [8] (a stochastic version of [9]) a theory of conserved quantities related to a stochastic differential equation of Stratonovich type has been presented, without referring to either Lagrangians or Hamiltonians. In this paper we investigate the symmetry of the stochastic dynamical differential equation and the space of conserved quantities. We derive new results on conserved quantities which include the ones in [8]. It is shown that the conserved quantities are related to the symmetry algebra of the space of conserved quantities.

We consider the stochastic dynamical systems of Stratonovich type [10] described by the following n-dimensional vector-valued stochastic differential equations:

$$\mathrm{d}x_t = b(x_t, t)\mathrm{d}t + \sum_{r=1}^{m} g_r(x_t, t) \circ \mathrm{d}w_t^r \tag{1}$$

where x_t is a $\mathbb{R}^n$-valued stochastic process, $w_t = (w_t^r)_{r=1}^m$ is a $\mathbb{R}^m$-valued standard Wiener process, $b = (b^i)_{i=1}^n$ and $g_r = (g_r^i)_{i=1}^n$ are $\mathbb{R}^n$-valued smooth functions, $r = 1, \ldots, m$, satisfying restrictions at infinity allowing the existence and uniqueness of solutions of (1), with given (deterministic or stochastic) initial condition $x_t|_{t=0} = x_0$. Let $\mathcal{F} \equiv C^2(\mathbb{R}^n \times \mathbb{R})$. A function $I \in \mathcal{F}$ is called a conserved quantity of a stochastic dynamical system (1) if it satisfies

$$\Delta_t I(x_t, t) = 0 \qquad \tilde{\Delta}_r I(x_t, t) = 0 \quad r = 1, \ldots, m \tag{2}$$

where $\Delta_t = \partial_t + \sum_{i=1}^n b^i \partial_i$ and $\tilde{\Delta}_r = \sum_{i=1}^n g_r^i \partial_i$, when x_t satisfies (1). By Ito–Stratonovich's formula equation (2) implies that $\mathrm{d}I(x_t, t) = 0$ and $I(x_t, t) =$ constant holds along the

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0305-4470/95/226363+09$19.50

diffusion process x_t, the constant being independent of t, but possibly depending on the initial condition x_0. If the initial condition x_0 in (1) is taken to be deterministic, then I is a deterministic quantity independent of time.

In investigating the symmetry of the stochastic dynamical process (1), we would like to distinguish between the symmetry of the stochastic differential equation (1) and the symmetry of the space of conserved quantities. We first consider the former. Let $\epsilon > 0$ and $\zeta = (\zeta)_{i=1}^n \in \mathcal{F}$.

Theorem 1. For ϵ sufficiently small the stochastic differential equation (1) is invariant under the following transformations:

$$x_t^i \to x_t^i + \epsilon\zeta^i(x_t, t) \qquad i = 1, \ldots, n \tag{3}$$

if the $\zeta^i(x_t, t)$ satisfy

$$\begin{aligned} &\Delta_t \zeta^i(x_t, t) - \sum_{j=1}^n \zeta^j(x_t, t)\partial_j b^i(x_t, t) = 0 \\ &\tilde{\Delta}_r \zeta^i(x_t, t) - \sum_{j=1}^n \zeta^j(x_t, t)\partial_j g_r^i(x_t, t) = 0 \qquad r = 1, \ldots, m. \end{aligned} \tag{4}$$

Proof. Under (3) equation (1) becomes (writing ζ as a shorthand for $\zeta(x_t, t)$)

$$\begin{aligned} \mathrm{d}(x_t^i + \epsilon\zeta^i) = \mathrm{d}x_t^i + \epsilon \mathrm{d}\zeta^i &= b^i(x_t + \epsilon\zeta, t)\mathrm{d}t + \sum_{r=1}^m g_r^i(x_t + \epsilon\zeta, t) \circ \mathrm{d}w_t^r \\ &= b^i(x_t, t)\mathrm{d}t + \sum_{j=1}^n \epsilon\zeta^j \partial_j b^i(x_t, t)\mathrm{d}t + \sum_{r=1}^m g_r^i(x_t, t) \circ \mathrm{d}w_t^r \\ &\quad + \sum_{r=1}^m \sum_{j=1}^n \epsilon\zeta^j \partial_j g_r^i(x_t, t) \circ \mathrm{d}w_t^r + o(\epsilon) \end{aligned}$$

with $\epsilon^{-1} o(\epsilon) \to 0$ as $\epsilon \downarrow 0$. That is

$$\mathrm{d}\zeta^i = \sum_{j=1}^n \zeta^j \partial_j b^i(x_t, t)\mathrm{d}t + \sum_{r=1}^m \sum_{j=1}^n \zeta^j \partial_j g_r^i(x_t, t) \circ \mathrm{d}w_t^r .$$

On the other hand, by the formula for Stratonovich differentials we have

$$\mathrm{d}\zeta^i = \Delta_t \zeta^i \mathrm{d}t + \sum_{r=1}^m \tilde{\Delta}_r \zeta^i \circ \mathrm{d}w_t^r .$$

Combining the above two equations we obtain equations (4). □

Let $a^i, i = 1, \ldots, n$ and a_0 all belong to $\mathcal{F}$. Then an operator $S = \sum_{i=1}^n a^i \partial_i + a_0 \partial_t$ (acting on $C^1(\mathbb{R}^n \times \mathbb{R})$-functions) is by definition a symmetry operator of the infinitesimal invariance transformation (3) of the stochastic equation (1) if S satisfies, on $C^1(\mathbb{R}^n \times \mathbb{R})$

$$[S, x_t^i] = \zeta^i = a^i \tag{5}$$

where [,] is the Lie bracket and ζ^i satisfies equation (4).

For the symmetry related to the space of conserved quantities of the stochastic dynamical process (1), we consider a linear operator L satisfying the following commutation relations on $C^1(\mathbb{R}^n \times \mathbb{R})$:

$$[\Delta_t, L] = T\Delta_t + \sum_{\alpha=1}^{m} T^\alpha \tilde{\Delta}_\alpha \qquad [\tilde{\Delta}_r, L] = R_r\Delta_t + \sum_{\alpha=1}^{m} R_r^\alpha \tilde{\Delta}_\alpha \quad r = 1, \ldots, m \tag{6}$$

where $T, T^\alpha, R_r, R_r^\alpha \in \mathcal{F}$. Let $\mathcal{I} \equiv \{I(x_t, t)|\mathrm{d}I(x_t, t) = 0$ when x_t satisfies (1)$\}$ be the space of the conserved functionals of the process. We have:

Theorem 2. For $I \in \mathcal{I}$ and L satisfying relation (6), LI is also a conserved quantity, i.e. $LI \in \mathcal{I}$.

Proof. As $I \in \mathcal{I}$, I satisfies equation (2). From (6) we further have $\Delta_t(LI) = 0$, $\tilde{\Delta}_r(LI) = 0$, $r = 1, \ldots, m$. Hence $LI \in \mathcal{I}$. □

Let $\mathcal{L}$ denote the set of all operators L satisfying (6).

Theorem 3. The set $\mathcal{L}$ is a complex Lie algebra under Lie commutators (acting on $C^1(\mathbb{R}^n \times \mathbb{R})$); that is, if $L_1, L_2, L_3 \in \mathcal{L}$, then:
(i) $a_1L_1 + a_2L_2 \in \mathcal{L}$, $\forall\, a_1, a_2 \in \mathbb{C}\backslash\{0\}$,
(ii) $[L_1, L_2] \in \mathcal{L}$,
(iii) $[L_1, [L_2, L_3]] + [L_2, [L_3, L_1]] + [L_3, [L_1, L_2]] = 0$.

Proof. Let $L_1, L_2 \in \mathcal{L}$, s.t., on $C^1(\mathbb{R}^n \times \mathbb{R})$:

$$[\Delta_t, L_i] = T_i\Delta_t + \sum_{\alpha=1}^{m} T_i^\alpha \tilde{\Delta}_\alpha \qquad [\tilde{\Delta}_r, L_i] = R_r^i\Delta_t + \sum_{\alpha=1}^{m} R_r^{i\alpha} \tilde{\Delta}_\alpha \qquad i = 1, 2.$$

Property (i) is obviously as

$$[\Delta_t, a_1L_1 + a_2L_2] = (a_1T_1 + a_2T_2)\Delta_t + \sum_{\alpha=1}^{m}(a_1T_1^\alpha a_1 + a_2T_2^\alpha)\tilde{\Delta}_\alpha$$

and

$$[\tilde{\Delta}_r, a_1L_1 + a_2L_2] = (a_1R_r^1 + a_2R_r^2)\Delta_t + \sum_{\alpha=1}^{m}(a_1R_r^{1\alpha} + a_2R_r^{2\alpha})\tilde{\Delta}_\alpha .$$

By a direct calculation we have, on $C^1(\mathbb{R}^n \times \mathbb{R})$:

$$[\Delta_t, [L_1, L_2]] = \left(L^1T_2 - L^2T_1 + \sum_{\alpha=1}^{m}(T_1^\alpha R_\alpha^2 - T_2^\alpha R_\alpha^1)\right)\Delta_t$$
$$+ \sum_{\alpha=1}^{m}\left(L_1T_2^\alpha - L_2T_1^\alpha + T_1T_2^\alpha - T_2T_1^\alpha + \sum_{\beta=1}^{m}(T_1^\beta R_\beta^{2\alpha} - T_2^\beta R_\beta^{1\alpha})\right)\tilde{\Delta}_\alpha$$

and

$$[\tilde{\Delta}_r, [L_1, L_2]] = \left(L^1R_r^2 - L^2R_r^1 + \sum_{\alpha=1}^{m}(R_r^{1\alpha}R_\alpha^2 - R_r^{2\alpha}R_\alpha^1)\right)\Delta_t$$
$$+ \sum_{\alpha=1}^{m}\left(L_1R_r^{2\alpha} - L_2R_r^{1\alpha} + R_r^1T_2^\alpha - R_r^2T_1^\alpha + \sum_{\beta=1}^{m}(R_r^{1\beta}R_\beta^{2\alpha} - R_r^{2\beta}R_\beta^{1\alpha})\right)\tilde{\Delta}_\alpha$$

where the linear property of the operators L_1 and L_2 has been used. Therefore we get $[L_1, L_2] \in \mathcal{L}$. The Jacobi identity (iii) is satisfied as $L \in \mathcal{L}$ are linear differential operators. □

From Theorem 2 we have that the space of the conserved functionals admits the symmetry algebra $\mathcal{L}$ in the sense that it is invariant under any $L \in \mathcal{L}$. The space $\mathcal{I}$ is a representation of the closed algebra $\mathcal{L}$. We call the elements of $\mathcal{L}$ 'symmetry operators'.

Now we consider a subalgebra $\mathcal{L}_0$ of $\mathcal{L}$ with $T^r = R_r = 0$, $R_r^\alpha = 0$, for $\alpha \neq r$ and $R_r^r = T$ in relation (6). That is, for $L_0 \in \mathcal{L}_0$, on $C^1(\mathbb{R}^n \times \mathbb{R})$

$$[\Delta_t, L_0] = T\Delta_t \qquad [\tilde{\Delta}_r, L_0] = T\tilde{\Delta}_r \quad r = 1, \ldots, m. \tag{7}$$

Let L_0 be of the form $L_0 = \sum_{i=1}^n A^i \partial_i + B\partial_t$ with $A^i, i = 1, \ldots, n$ and B belong to $\mathcal{F}$. A direct calculation shows that relations (7) are equivalent to the following equations:

$$\Delta_t B = T \tag{8}$$

$$\Delta_t A^i - \sum_{j=1}^n A^j \partial_j b^i - B\partial_t b^i - Tb^i = 0 \tag{9}$$

and

$$\tilde{\Delta}_r B = 0 \qquad r = 1, \ldots, m \tag{10}$$

$$\tilde{\Delta}_r A^i - \sum_{j=1}^n A^j \partial_j g_r^i - B\partial_t g_r^i - Tg_r^i = 0 \tag{11}$$

as operators on $C^1(\mathbb{R}^n \times \mathbb{R})$.

We remark that in general a symmetry operator of the space $\mathcal{I}$ is not a symmetry operator of the infinitesimal transformations of the stochastic differential equation. But when $B = 0$, then the equation set (8)–(11) reduces to the equation set (4) by replacing A^i with $a^i(= \zeta^i)$ and $\sum_{i=1}^n A^i \partial_i$ is both a symmetry operator of the infinitesimal invariance transformation of the stochastic differential equation and for the space $\mathcal{I}$ of conserved quantities.

Theorem 4. Let $L_0 = \sum_{i=1}^n A^i \partial_i + B\partial_t \in \mathcal{L}_0$. Then

$$I(x_t, t) = \sum_{i=1}^n \partial_i A^i(x_t, t) + \partial_t B(x_t, t) - T(x_t, t) + L_0\phi(x_t, t) \tag{12}$$

is a conserved quantity of stochastic dynamical system (1) (i.e. for x_t satisfies (1)), when $\phi \in \mathcal{F}$ satisfies

$$\tilde{\Delta}_r \phi(x_t, t) + \sum_{i=1}^n \partial_i g_r^i(x_t, t) = 0 \qquad r = 1, \ldots, m \tag{13}$$

$$\Delta_t \phi(x_t, t) + \sum_{i=1}^n \partial_i b^i(x_t, t) = 0. \tag{14}$$

Proof. We have (dropping everywhere, for simplicity, the arguments x_t, t)

$$\Delta_t I = \Delta_t\left(\sum_{i=1}^{n}\partial_i A^i\right) + \Delta_t(L_0\phi) + \Delta_t(\partial_t B - T)$$

$$= \sum_{i=1}^{n}\partial_i\left(\Delta_t A^i\right) - \sum_{i,j=1}^{n}\partial_j b^i\partial_i A^j + \Delta_t(L_0\phi) + \Delta_t(\partial_t B - T)$$

$$= \sum_{i,j=1}^{n}A^j\partial_j\partial_i b^i + B\sum_{i=1}^{n}\partial_t\partial_i b^i + \sum_{i=1}^{n}\partial_i B\partial_t b^i + T\sum_{i=1}^{n}\partial_i b^i$$

$$+\sum_{i=1}^{n}b^i\partial_i T + T\Delta_t\phi + L_0\Delta_t\phi + \Delta_t(\partial_t B - T)$$

$$= (L_0 + T)\left(\sum_{i=1}^{n}\partial_i b^i + \Delta_t\phi\right) = 0$$

$$\tilde{\Delta}_r I = \tilde{\Delta}_r\left(\sum_{i=1}^{n}\partial_i A^i\right) + \tilde{\Delta}_r(L_0\phi) + \tilde{\Delta}_r(\partial_t B - T)$$

$$= \sum_{i=1}^{n}\partial_i(\tilde{\Delta}_r A^i) - \sum_{i,j=1}^{n}\partial_j g_r^i\partial_i A^j + \tilde{\Delta}_r(L_0\phi) + \tilde{\Delta}_r(\partial_t B - T)$$

$$= \sum_{i,j=1}^{n}A^j\partial_j\partial_i g_r^i + B\sum_{i=1}^{n}\partial_t\partial_i g_r^i + \sum_{i=1}^{n}\partial_i B\partial_t g_r^i + T\sum_{i=1}^{n}\partial_i g_r^i$$

$$+\sum_{i=1}^{n}g_r^i\partial_i T + T\tilde{\Delta}_r\phi + L_0\tilde{\Delta}_r\phi + \tilde{\Delta}_r(\partial_t B - T)$$

$$= (L_0 + T)\left(\sum_{i=1}^{n}\partial_i g_r^i + \tilde{\Delta}_r\phi\right) = 0$$

where equations (8)–(11), (13) and (14) have been used. Therefore by definition $I(x_t, t)$ is a conserved quantity. □

Theorem 4 is a generalization of that presented in [8], not only because of the extra term $\partial_t B - T$, but also because of the presence of B in the symmetry operator L_0. For the special case that $b^i = g_r^i$, or more generally $g_r^i = C(x_t, t)b^i$, $r = 1, \ldots, m$, $\forall\ C(x_t, t) \in \mathcal{F}$ (these are the cases of the examples given in [8]), we see from (8) and (10) that $\partial_t B - T = 0$, hence these terms disappear in the expression of $I(x_t, t)$. Even in these cases equation (12) is still a generalization of that in [8] as long as $B \neq 0$ in L_0.

For a more detailed discussion we consider several examples.

Example 1. Following [8] we consider the three-dimensional stochastic linear dynamical system

$$\mathrm{d}\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{pmatrix} x_t^3 - x_t^2 \\ x_t^1 - x_t^3 \\ x_t^2 - x_t^1 \end{pmatrix}\mathrm{d}t + \begin{pmatrix} x_t^3 - x_t^2 \\ x_t^1 - x_t^3 \\ x_t^2 - x_t^1 \end{pmatrix} \circ \mathrm{d}w_t\,. \tag{15}$$

In this case the existence and uniqueness of the solution is well known (see, e.g., [10]). The system (15) has the properties $g^i = b^i$, $\sum_{i=1}^3 \partial_i g^i = 0$ and $\sum_{i=1}^3 \partial_i b^i = 0$. From equations (8)–(11) several solutions of L_0 satisfying (7) can be obtained with $T = 0$. For instance

$$L_0 = (x_t^1 + x_t^2 + x_t^3) \sum_{i=1}^3 \partial_i$$

$$L_1 = \left[(x_t^1)^2 + (x_t^2)^2 + (x_t^3)^2\right] \sum_{i=1}^3 \partial_i$$

$$L_2 = (x_t^1 x_t^2 + x_t^2 x_t^3 + x_t^3 x_t^1) \sum_{i=1}^3 \partial_i$$

$$L_3 = \left[(x_t^1)^2(x_t^2 + x_t^3) + (x_t^2)^2(x_t^1 + x_t^3) + (x_t^3)^2(x_t^2 + x_t^1) + 3x_t^1 x_t^2 x_t^3\right] \sum_{i=1}^3 \partial_i .$$

By using theorem 4 we can deduce that the following quantities are conserved:

$$I_0 = \text{constant (independent of the } x_t^i)$$

$$I_1 = I_2 = x_t^1 + x_t^2 + x_t^3$$

$$I_3 = 2((x_t^1)^2 + (x_t^2)^2 + (x_t^3)^2) + 7(x_t^1 x_t^2 + x_t^2 x_t^3 + x_t^3 x_t^1)$$

where I_1 is the conserved quantity obtained in [8]. I_3 is a new conserved quantity for the system (15).

As Δ_t and $\tilde{\Delta}_r$ are linear operators, products of conserved quantities are still conserved quantities. Let us set

$$I_3' \equiv (I_3 - 2I_2^2)/3 = x_t^1 x_t^2 + x_t^2 x_t^3 + x_t^3 x_t^1 .$$

I_1 and I_3' are then two simple non-trivial conserved quantities of the system (15). Symmetry operators map conserved quantities into conserved quantities. Under the actions of the symmetry operators L_i, $i = 1, 2, \ldots$, we have, e.g.,

$$L_0 I_1 = 3I_1 \qquad L_1 I_1 = 3(I_1^2 - 2I_3')$$

$$L_2 I_1 = 3I_3' \qquad L_0 I_3' = 2I_1^2 .$$

In fact $L_0 = I_1 \sum_{i=1}^3 \partial_i$ and $L_2 = I_3' \sum_{i=1}^3 \partial_i$. In the present case $r = 1$ and

$$\left[\sum_{i=1}^3 \partial_i, \Delta_t\right] = 0 \qquad \left[\sum_{i=1}^3 \partial_i, \tilde{\Delta}_1\right] = 0$$

on $C^1(\mathbb{R}^3 \times \mathbb{R})$ functions. Let f be an arbitrary polynomial function on $\mathbb{R}^2$. Since $L \equiv f(I_1, I_3') \sum_{i=1}^3 \partial_i$ commutes with Δ_t and $\tilde{\Delta}_1$, we have that L is a symmetry operator in $\mathcal{L}_0$ (defined in (7)). Hence the system (15) possesses an infinite number of symmetry operators that are linearly independent. They constitute an algebra $\mathcal{L}_0$ with commutation relations which can obviously be explicitly computed, e.g.,

$$[L_0, L_1] = 4L_3 - L_1 \qquad [L_0, L_2] = 2L_1 + L_3$$

on $C^1(\mathbb{R}^n \times \mathbb{R})$. As $B = 0$ in this example, the algebra $\mathcal{L}_0$ coincides with the algebra generating the infinitesimal invariance transformations for the stochastic differential equation.

The following examples are designed to show the useful symmetry analysis on conserved functionals in stochastic dynamical systems. We start with a (unique) solution for small times and show that there are conserved quantities (functionals of the solution process) associated with it.

Example 2. Let us consider the following stochastic dynamical system:

$$\mathrm{d}x_t^i = b^i(x_t, t)\mathrm{d}t + g^i(x_t, t) \circ \mathrm{d}w_t \tag{16}$$

with

$$g^i = x_t^i((x_t^1)^2 + (x_t^2)^2 + (x_t^3)^2)^m \left(\mathrm{e}^{2mt}\delta_{n,0} - \frac{1}{nt}\delta_{n,-2m}\right)$$

$$b^i = \frac{x_t^i}{(x_t^1 + x_t^2 + x_t^3)^n} (\mathrm{e}^{2mt}\delta_{n,0} - b_0\delta_{n,-2m}) \qquad i = 1, 2, 3$$

where $n, m \in Z$, $m \neq 0$, and $b_0 \in \mathbb{R}$. We have the symmetry operators in $\mathcal{L}_0$ satisfying (on $C^1(\mathbb{R}^n \times \mathbb{R})$)

$$[\Delta_t, L_i] = T_i\Delta_t \qquad [\tilde{\Delta}_1, L_i] = T_i\tilde{\Delta}_1 \qquad i = 1, 2, 3,$$

with

$$\begin{aligned}
L_1 &= x_t^3\partial_2 - x_t^2\partial_3 + \frac{x_t^2 - x_t^3}{x_t^1 + x_t^2 + x_t^3}(\mathrm{e}^{-2mt}\delta_{n,0} - nt\delta_{n,-2m})\partial_t \\
L_2 &= x_t^1\partial_3 - x_t^3\partial_1 + \frac{x_t^3 - x_t^1}{x_t^1 + x_t^2 + x_t^3}(\mathrm{e}^{-2mt}\delta_{n,0} - nt\delta_{n,-2m})\partial_t \\
L_3 &= x_t^2\partial_2 - x_t^1\partial_2 + \frac{x_t^1 - x_t^2}{x_t^1 + x_t^2 + x_t^3}(\mathrm{e}^{-2mt}\delta_{n,0} - nt\delta_{n,-2m})\partial_t
\end{aligned} \tag{17}$$

and

$$T_1 = \frac{2m(x_t^3 - x_t^2)}{x_t^1 + x_t^2 + x_t^3}(\mathrm{e}^{-2mt}\delta_{n,0} - \delta_{n,-2m})$$

$$T_2 = \frac{2m(x_t^1 - x_t^3)}{x_t^1 + x_t^2 + x_t^3}(\mathrm{e}^{-2mt}\delta_{n,0} - \delta_{n,-2m})$$

$$T_3 = \frac{2m(x_t^2 - x_t^1)}{x_t^1 + x_t^2 + x_t^3}(\mathrm{e}^{-2mt}\delta_{n,0} - \delta_{n,-2m})$$

where $\Delta_t = \sum_{i=1}^3 b^i\partial_i + \partial_t$ and $\tilde{\Delta}_1 = \sum_{i=1}^3 g^i\partial_i$.

A function ϕ satisfying (13) is given by

$$\phi(x_t, t) = -\frac{3 + 2m}{3}\log(x_t^1x_t^2x_t^3) + \mathrm{e}^{2mt}\delta_{n,0}$$

for $x_t^i \neq 0$, $i = 1, 2, 3$. From theorem 4 we have the conserved quantities

$$I_1 = \frac{(2m+3)((x_t^2)^2 - (x_t^3)^2)}{3x_t^2x_t^3} + \frac{2m(x_t^2 - x_t^3)}{x_t^1 + x_t^2 + x_t^3}\delta_{n,0}$$

$$I_2 = \frac{(2m+3)((x_t^3)^2 - (x_t^1)^2)}{3x_t^1x_t^3} + \frac{2m(x_t^3 - x_t^1)}{x_t^1 + x_t^2 + x_t^3}\delta_{n,0}$$

$$I_3 = \frac{(2m+3)((x_t^1)^2 - (x_t^2)^2)}{3x_t^1x_t^2} + \frac{2m(x_t^1 - x_t^2)}{x_t^1 + x_t^2 + x_t^3}\delta_{n,0}.$$

In this example the terms $\partial_t B_i - T_i,\ i = 1, 2, 3$ appearing in (12) are zero. But as $B_i \neq 0$, the term $L_i\phi$ still contributes extra terms to I_i.

The space of conserved quantities of the system (16) is $SU(2)$ symmetric. It is straightforward to check that the symmetry operators (17) satisfy the $SU(2)$ algebraic relations

$$[L_i,\ L_j] = \epsilon_{ijk} L_k \qquad i, j, k = 1, 2, 3.$$

Example 3. The following example is a nonlinear model with $\partial_t B - T \neq 0$ in (12):

$$\mathrm{d}\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \frac{1}{t}\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix}\mathrm{d}t + \frac{1}{t}\begin{pmatrix} x_t^1(x_t^3 - x_t^2) \\ x_t^2(x_t^1 - x_t^3) \\ x_t^3(x_t^2 - x_t^1) \end{pmatrix} \circ \mathrm{d}w_t \qquad t > 0. \tag{18}$$

For this system we have a symmetry operator $L_0 \in \mathcal{L}_0$ given by

$$L_0 = B\partial_t = (x_t^1 + x_t^2 + x_t^3)\partial_t$$

satisfying

$$[\Delta_t, L_0] = T\Delta_t = \frac{1}{t}(x_t^1 + x_t^2 + x_t^3)\Delta_t$$

$$[\tilde{\Delta}_1, L_0] = T\tilde{\Delta}_1 = \frac{1}{t}(x_t^1 + x_t^2 + x_t^3)\tilde{\Delta}_1$$

where

$$\Delta_t = \partial_t + \sum_{i=1}^{3} \frac{x_t^i}{t}\partial_i \qquad \tilde{\Delta}_1 = \frac{x_t^1(x_t^3 - x_t^2)}{t}\partial_1 + \frac{x_t^2(x_t^1 - x_t^3)}{t}\partial_2 + \frac{x_t^3(x_t^2 - x_t^1)}{t}\partial_3. \tag{19}$$

ϕ satisfying (13) and (14) is given by $\phi = -3\log t,\ t \neq 0$. From theorem 4 we have, for x_t satisfying (18),

$$I(x_t, t) = \sum_{i=1}^{n} \partial_i A^i + \partial_t B - T + L_0\phi = -\frac{4}{t}(x_t^1 + x_t^2 + x_t^3) \qquad t > 0.$$

Let us summarize the above. By investigating the symmetry of the space of conserved quantities for stochastic dynamical systems, we have established new relations for conserved quantities. We would like to indicate that although the conserved functionals are given by the elements of a subalgebra $\mathcal{L}_0$ of $\mathcal{L}$, the space of conserved functionals itself admits the symmetry Lie algebra $\mathcal{L}$. Let us consider (18), with Δ_t and $\tilde{\Delta}_1$ given by (19), as an example. We consider the operator $L = a(x_t, t)\Delta_t + b(x_t, t)\tilde{\Delta}_1$, $a(x_t, t), b(x_t, t) \in \mathcal{F}$. Noting that in present case $[\Delta_t, \tilde{\Delta}_1] = 0$, we have

$$[\Delta_t, L] = \Delta_t a(x_t, t)\Delta_t + \Delta_t b(x_t, t)\tilde{\Delta}_1 \qquad [\tilde{\Delta}_1, L] = \tilde{\Delta}_1 a(x_t, t)\Delta_t + \tilde{\Delta}_1 b(x_t, t)\tilde{\Delta}_1.$$

Therefore L is a symmetry operator in $\mathcal{L}$ which maps $I \in \mathcal{I}$ to zero. However only when $\Delta_t b(x_t, t) = \tilde{\Delta}_1 a(x_t, t) = 0$ and $\tilde{\Delta}_1 b(x_t, t) = \Delta_t a(x_t, t)$ is L a symmetry operator in $\mathcal{L}_0$, satisfying the defining relations (7).

Acknowledgment

S-MF thanks the Alexander von Humboldt Foundation for financial support.

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